

Amalgamated Products of Ore and Quadratic Extensions of Rings

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ABSTRACT. We study the ideal theory of amalgamated products of Ore and quadratic extensions over a base ring R . We prove an analogue of the Hilbert Basis theorem for an amalgamated product Q of quadratic extensions and determine conditions for when the one-sided ideals of Q are principal or doubly-generated. We also determine conditions that make Q a principal ideal ring. Finally, we show that the double affine Hecke algebra $\mathbb{H}_{q,t}$ associated to the general linear group $GL_2(k)$ (here, k is a field with $\text{char}(k) \neq 2$) is an amalgamated product of quadratic extensions over the three-dimensional quantum torus $\mathcal{O}_q((k^\times)^3)$ and give an explicit isomorphism. In this case, it follows that $\mathbb{H}_{q,t}$ is a noetherian ring.

1. Introduction

Since the appearance of the seminal paper of Ore [7] in 1933, Ore extensions $R[x; \tau, \delta]$ (or *skew-polynomial rings*) have played an important roll in several areas of algebra, such as the universal enveloping algebras of solvable Lie algebras, group rings of polycyclic-by-finite groups, quantized coordinate rings, and rings of differential operators. The Ore extensions of a given ring R behave like an ordinary polynomial ring $R[x]$, except the coefficients do not necessarily commute past the variable x . The noncommutativity is governed by a ring endomorphism $\tau : R \rightarrow R$ and a (left) τ -derivation $\delta : R \rightarrow R$. In an Ore extension based on these data $xr = \tau(r)x + \delta(r)$ ($r \in R$).

Let $S_1 = R[x; \tau_1, \delta_1]$ and $S_2 = R[y; \tau_2, \delta_2]$ be Ore extensions over a base ring R , and let $S := S_1 *_R S_2$, the pushout (or *amalgamated free product*) ring. Primeness of the base ring R carries over to the amalgamated product S . In Section 2.1 we prove

THEOREM 1.1. *Let τ_1 and τ_2 be automorphisms of a ring R . If R is a (τ_1, τ_2) -prime ring (i.e. if the product of two (τ_1, τ_2) -stable ideals is zero, then at least one ideal is zero), then S is a prime ring.*

Since the notion of (τ_1, τ_2) -prime is weaker than the usual notion of prime, it immediately follows that if R is prime, then S is also prime (Theorem 2.5). Secondly, we show a prime $(\tau_1, \tau_2, \delta_1, \delta_2)$ -ideal I in a noetherian ring R will generate a prime ideal in S , and the factor ring S/I is isomorphic to the amalgamated product $(R/I)[x, \tau_1, \delta_1] *_R (R/I)[y, \tau_2, \delta_2]$. Certain localizations of S reduce to studying the case where the base ring R is a division ring.

THEOREM 1.2. *Let τ_1 and τ_2 be automorphisms of a noetherian domain R , and let $X := R \setminus \{0\}$. Then*

- (i) X is a right denominator set in S , and
- (ii) the ring of fractions SX^{-1} is isomorphic to the amalgamated product

$$(RX^{-1})[x; \tau_1, \delta_1] *_R (RX^{-1})[y; \tau_2, \delta_2],$$

where $\tau_1, \tau_2, \delta_1, \delta_2$ denote the induced maps on the division ring of fractions RX^{-1} .

In Section 2.2, we study the amalgamated product $Q_1 *_R Q_2$ of quadratic extensions of a base ring R . We call an extension of rings $R \subseteq A$ *quadratic* if there exists $x \in A \setminus R$ so that A is a free left R -module with basis $\{1, x\}$. A quadratic extension of R will necessarily be isomorphic to a factor ring of an Ore extension

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$R[x; \tau, \delta]$. More precisely, $A \cong R[x; \tau, \delta]/I$, where I is a principal ideal in $R[x; \tau, \delta]$ generated by an element of the form $x^2 + ax + b$ ($a, b \in R$). Let $Q_1 = S_1/I_1$ and $Q_2 = S_2/I_2$ be quadratic extensions over R , and put $Q = Q_1 *_R Q_2$. Under certain conditions, the ideal theory of Q turns out to have a nice description. In particular, we have an analogue of the Hilbert Basis theorem.

THEOREM 1.3. *If τ_1 and τ_2 are automorphisms of a noetherian base ring R , then Q is noetherian.*

Furthermore, we prove the following.

THEOREM 1.4. *If the base ring R is a division ring and τ_1 and τ_2 are automorphisms, then all one-sided ideals of Q are either principal or doubly generated.*

Theorem 1.4 above is reminiscent of a famous result of Stafford. In [8], Stafford proves the Weyl algebras $A_n(k) := k \left\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle$ over a field k of characteristic zero exhibit this property: every one-sided ideal of $A_n(k)$ is principal or doubly generated. Placing extra restrictions on the base ring R and the skew derivations δ_1, δ_2 forces the amalgamated product Q to be a principal ideal ring.

THEOREM 1.5. *If τ_1 and τ_2 are automorphisms of a division ring R and the skew derivations δ_1, δ_2 are not τ_1 (resp τ_2)-inner, then Q is a principal ideal ring.*

One application to studying these types of extensions and their amalgamated products comes from the theory of double affine Hecke algebras. The double affine Hecke algebras are algebras related to symmetric polynomials and were introduced by Cherednik in the early 1990's [1]. They were instrumental to the proof of the Macdonald constant-term conjectures [6].

For a reductive algebraic group G with rank n over a field k , the DAHA $\mathbb{H}_{q,t}(G)$ associated to G is a deformation (with deformation parameters $q_1, \dots, q_n, t_1, \dots, t_n$) of the group algebra of the *extended double affine Weyl group* $W \ltimes (P \oplus P)$, where W is the Weyl group of G and P is the weight lattice. When the deformation parameters are all specialized to $1 \in k$, we recover the group algebra $k[W \ltimes (P \oplus P)]$.

In [3], Gehles studies the structure theory of the DAHAs and their associated trigonometric and rational degenerations and proves the DAHAs are noetherian when the deformation parameters q_1, \dots, q_n are specialized to $1 \in k$ (but the t_i 's may be arbitrarily chosen) [3, Corollary 2.1.9]. It remains an open problem to prove noetherianity for arbitrary deformation parameters.

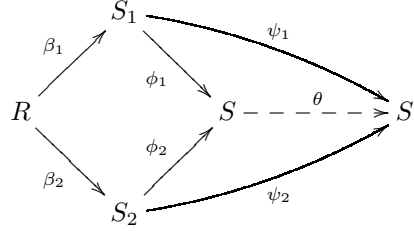
Let k be a field with $\text{char}(k) \neq 2$ and let $\mathcal{O}_{\mathbf{q}}((k^\times)^3)$ denote the k -algebra generated by invertible variables z_1, z_2, z_3 and subject to the defining relations $z_1 z_2 = z_2 z_1$, $z_1 z_3 = q^{-1} z_3 z_1$, and $z_2 z_3 = q^{-1} z_3 z_2$. The ring $\mathcal{O}_{\mathbf{q}}((k^\times)^3)$ is commonly referred to as (the quantized coordinate ring of) a three-dimension quantum torus. In Section 3, we prove $\mathbb{H}_{q,t}(GL_2(k))$ is isomorphic to an amalgamated product of quadratic extensions over $\mathcal{O}_{\mathbf{q}}((k^\times)^3)$. In Theorem 3.1, we give an explicit isomorphism. Here, the deformation parameters q and t may be chosen to be arbitrary nonzero scalars in k such that $t^{1/2}$ exists. Since $\mathcal{O}_{\mathbf{q}}((k^\times)^3)$ is known to be noetherian, it follows that $\mathbb{H}_{q,t}(GL_2(k))$ is also noetherian.

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2. Amalgamated Products of Rings

In this section, we prove some general results concerning the structure theory of amalgamated products of quadratic extensions and Ore extensions of rings. The results are then applied to the example of the double affine Hecke algebra $\mathbb{H}_{q,t}(GL_2(k))$.

DEFINITION 2.1. *Let $\beta_1 : R \rightarrow S_1$ and $\beta_2 : R \rightarrow S_2$ be ring homomorphisms. The amalgamated product of S_1 and S_2 along R is a triple (S, ϕ_1, ϕ_2) where $\phi_1 : S_1 \rightarrow S$ and $\phi_2 : S_2 \rightarrow S$ are ring homomorphisms satisfying $\phi_1 \beta_1 = \phi_2 \beta_2$ and the universal property: Given any ring S' with homomorphisms $\psi_1 : S_1 \rightarrow S'$, $\psi_2 : S_2 \rightarrow S'$ satisfying $\psi_1 \beta_1 = \psi_2 \beta_2$, there exists a unique homomorphism $\theta : S \rightarrow S'$ so that the following diagram commutes.*



In all examples we consider, R is a subring of S_i ($i = 1, 2$) and $\beta_i : R \rightarrow S_i$ are inclusion maps. When referring to an amalgamated product, we will usually not mention the ring R and simply say S is an amalgamated product of S_1 and S_2 . It is well-known that an amalgamated product S of S_1 and S_2 exists and is unique up to isomorphism. Thus, we will call S *the* amalgamated product and denote it by $S = S_1 *_R S_2$. The presentation we will use for $S_1 *_R S_2$ is given by the generating set $R \sqcup S_1 \sqcup S_2$. There are two main types of relations among the generators. First of all, if r and s are both in R , S_1 , or S_2 , then their product in $S_1 *_R S_2$ is the same as their product in the appropriate ring. The other defining relations are the “cross-relations”: for every $r \in R$, $s_1 \in S_1$, and $s_2 \in S_2$,

- (i) $rs_1 :=$ the product $\beta_1(r) \cdot s_1$ in S_1 ,
- (ii) $s_1r :=$ the product $s_1 \cdot \beta_1(r)$ in S_1 ,
- (iii) $rs_2 :=$ the product $\beta_2(r) \cdot s_2$ in S_2 ,
- (iv) $s_2r :=$ the product $s_2 \cdot \beta_2(r)$ in S_2 ,
- (v) $r = \beta_1(r) = \beta_2(r)$.

2.1. Amalgamated Products of Ore extensions. One example we consider is the amalgamated product of Ore extensions over a base ring R . Recall, an *Ore extension over R* with left-hand coefficients is a ring A satisfying the following conditions:

- (i) A contains R as a subring,
- (ii) there exists $x \in A$ so that A is a free left R -module having basis $\{1, x, x^2, x^3, \dots\}$, and
- (iii) $xR \subseteq Rx + R$.

From the definition, it follows that there exists a ring endomorphism $\tau : R \rightarrow R$ and a left τ -derivation $\delta : R \rightarrow R$ (i.e. a \mathbb{Z} -linear map satisfying $\delta(rs) = \tau(r)\delta(s) + \delta(r)s$ for all $r, s \in R$) so that

$$(2.1) \quad xr = \tau(r)x + \delta(r)$$

for every $r \in R$. Given any such pair of maps (τ, δ) , a corresponding Ore extension exists and is unique up to isomorphism. We denote it by $R[x; \tau, \delta]$.

Throughout the remainder of this section, τ_1 and τ_2 will denote automorphisms of a ring R , and δ_1 and δ_2 are left τ_1 (resp. τ_2)-derivations of R .

We let $S_1 = R[x; \tau_1, \delta_1]$, $S_2 = R[y; \tau_2, \delta_2]$, and put $S = S_1 *_R S_2$. It follows from Eqn. 2.1 that the set of words in the letters x and y are a spanning set (over R) for S . (In fact, the set of words is an R -basis of S . One can verify R -linear independence by using the standard argument: let E be the semigroup ring over R of the free semigroup on two letters X and Y , viewed as a left R -module, and construct an S -module structure on E that mimics left multiplication in S . Finally consider the image of $1 \in E$ by the action of an arbitrary element $s \in S$.) There are two natural ways to order words, alphabetically and by length. Ordering first by length, then alphabetically (if lengths equal) gives us a total ordering on words. Thus, every nonzero $s \in S$ may be written in the form $s = rw + [\text{lower terms}]$ for some nonzero $r \in R$ and word w . We call rw the *leading term* of s , and r is the *leading coefficient*.

We need to recall the notion of η -stable ideals and establish some notation used in the results that follow. Let R^R be the set of all functions from a ring R to itself, and let \mathbb{X} be a subset of R . If $\eta \subseteq R^R$ and $f(\mathbb{X}) \subseteq \mathbb{X}$ for all $f \in \eta$, then \mathbb{X} is called *η -stable*. An ideal $I \subseteq R$ is called an *η -ideal* if I is η -stable. A proper ideal $P \subseteq R$ is *η -prime* if for any pair of η -ideals I, J of R with $IJ \subseteq P$, we have either $I \subseteq P$ or $J \subseteq P$. A ring R is called *η -prime* if 0 is an η -prime ideal of R . In what follows, we let $\mathbb{Z}_{\geq 0}^{fin.seq.}$ ($\mathbb{Z}^{fin.seq.}$) denote the set

of sequences in $\mathbb{Z}_{\geq 0}$ (and \mathbb{Z} resp.) having finitely many nonzero terms. For such a sequence $\mathbf{j} = (j_1, j_2, j_3, \dots)$, we use the notation $\tau^{\mathbf{j}}$ to mean the automorphism $\tau_1^{j_1} \tau_2^{j_2} \tau_1^{j_3} \tau_2^{j_4} \tau_1^{j_5} \dots$.

THEOREM 2.2. *If R is a domain, then S is a domain also.*

PROOF. Assume $s_1, s_2 \in S$ are nonzero. Therefore, we may write them in the form $s_1 = r_I w_I + [\text{lower terms}]$, $s_2 = r_J w_J + [\text{lower terms}]$, for some $r_I, r_J \in R$ nonzero. Thus,

$$s_1 s_2 = r_I r'_J w_I w_J + [\text{lower terms}],$$

where r'_J satisfies the identity $w_I r_J = r'_J w_I + [\text{lower terms}]$. Since $r'_J = \tau^{\mathbf{j}}(r_J)$ for some sequence \mathbf{j} that depends on w_I , this implies r'_J is nonzero. Hence the leading term of $s_1 s_2$ is nonzero. Thus, $s_1 s_2 \neq 0$. \square

The following lemma will be useful in the proof of Theorem 2.5.

LEMMA 2.3. *The following are equivalent:*

- (i) R is (τ_1, τ_2) -prime
- (ii) For all nonzero $r, r' \in R$, there exist $s \in R$ and a finite sequence $\mathbf{j} \in \mathbb{Z}^{fin.seq}$ so that $rs\tau^{\mathbf{j}}(r') \neq 0$.

PROOF. First, we assume that R is (τ_1, τ_2) -prime. Let $r, r' \in R$ be nonzero. The smallest (τ_1, τ_2) -ideals containing r and r' are

$$I_r = \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^{fin.seq}} R(\tau^{\mathbf{j}}(r))R \quad \text{and} \quad I_{r'} = \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^{fin.seq}} R(\tau^{\mathbf{j}}(r'))R$$

respectively. Since $I_r I_{r'} \neq 0$, we have $\tau^{\mathbf{j}}(r)r''(\tau^{\mathbf{j}'}(r')) \neq 0$ for some $r'' \in R$ and $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}_{\geq 0}^{fin.seq}$. Applying the inverse of $\tau^{\mathbf{j}}$ to both sides of $\tau^{\mathbf{j}}(r)r''(\tau^{\mathbf{j}'}(r')) \neq 0$ yields condition (ii). Conversely, assume that condition (ii) holds and let I, J be nonzero (τ_1, τ_2) -ideals of R . Pick $r \in I$ and $r' \in J$ both nonzero. There exist $r'' \in R$ and finite sequences $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}_{\geq 0}^{fin.seq}$ so that $\tau^{\mathbf{j}}(r)r''(\tau^{\mathbf{j}'}(r')) \neq 0$. Since $\tau^{\mathbf{j}}(r) \in I$ and $r''\tau^{\mathbf{j}'}(r') \in J$, the product IJ is nonzero. \square

THEOREM 2.4. *If R is (τ_1, τ_2) -prime, then S is prime.*

PROOF. Suppose $f, g \in S$ and there exist nonzero $r, s \in R$ so that $f = rw_I + [\text{lower terms}]$ and $g = sw_J + [\text{lower terms}]$. By Lemma 2.3, there exist $s' \in R$ and a sequence \mathbf{j} so that $rs'\tau^{\mathbf{j}}(s) \neq 0$. Furthermore, for every $r' \in R$ and word $w \in S$, we have

$$\begin{aligned} fr'wg &= (rw_I)(r'w)(sw_J) + [\text{lower terms}] \\ &= \underbrace{(rr'_I s_{w,I})}_{\in R} w_I w w_J + [\text{lower terms}], \end{aligned}$$

where $r'_I, s_{w,I} \in R$ satisfy $w_I r' = r'_I w_I + [\text{lower terms}]$ and $w_I w s = s_{w,I} w_I w + [\text{lower terms}]$. In particular, by choosing r' so that $r'_I = s'$ and w so that $s_{w,I} = \tau^{\mathbf{j}}(s)$, it follows that $fSg \neq 0$. Hence S is prime. \square

Since (τ_1, τ_2) -primeness of a ring is weaker than the usual notion of primeness, we have

THEOREM 2.5. *If R is prime, then S is prime also.*

The following theorem and the corollary that follows tell us when prime ideals of R generate prime ideals in S .

THEOREM 2.6. *If I is a $(\tau_1, \tau_2, \delta_1, \delta_2)$ -ideal of a noetherian ring R , then $IS = SI$ is an ideal of S and, letting τ_1, τ_2, δ_1 and δ_2 denote the induced functions on R/I , we have*

$$(2.2) \quad S/IS \cong (R/I)[x, \tau_1, \delta_1] *_{R/I} (R/I)[y, \tau_2, \delta_2].$$

PROOF. Since I is $(\tau_1, \tau_2, \delta_1, \delta_2)$ -stable, we have $xI \subseteq Ix + I$, and $yI \subseteq Iy + I$. Therefore $SI \subseteq IS + I \subseteq IS$. Next, we prove $SI \supseteq IS$. We have an ascending chain of ideals of R :

$$I \subseteq \tau_i^{-1}(I) \subseteq \tau_i^{-2}(I) \subseteq \tau_i^{-3}(I) \subseteq \dots$$

Thus, for some $N \in \mathbb{N}$ sufficiently large, $\tau_i^{-N}(I) = \tau_i^{-N-1}(I)$. Applying τ_i^N to both sides yields $I = \tau_i^{-1}(I)$. Since $rx = x\tau_1^{-1}(r) - \delta_1\tau_1^{-1}(r)$ and $ry = x\tau_2^{-1}(r) - \delta_2\tau_2^{-1}(r)$ for all $r \in R$, we have $Ix \subseteq xI + I$, and $Iy \subseteq yI + I$. Hence, $IS \subseteq SI + I \subseteq SI$. \square

COROLLARY 2.7. *If I is a prime $(\tau_1, \tau_2, \delta_1, \delta_2)$ -ideal of a noetherian ring R , then $IS = SI$ is a prime ideal of S and $S/IS \cong (R/I)[x, \tau_1, \delta_1] *_{R/I} (R/I)[y, \tau_2, \delta_2]$.*

Next we show that certain right denominator sets in the base ring R extend to right denominator sets in the amalgamated product S . Sometimes an appropriate localization of S will reduce to studying the case where the base ring R is a division ring.

First we recall that if X is a right denominator set in a ring R , then X is also a right denominator set in the Ore extension $R[x; \tau, \delta]$ provided τ is an automorphism and $\tau(X) = X$ (see e.g. [4, Lemma 1.4]). Furthermore, the identity map on the right ring of fractions RX^{-1} extends to an isomorphism of $R[x; \tau, \delta]X^{-1}$ onto $(RX^{-1})[x; \bar{\tau}, \bar{\delta}]$ sending $x1^{-1}$ to x , where $\bar{\tau}$ and $\bar{\delta}$ denote the induced maps on RX^{-1} [4, Lemma 1.4]. We have an analogous result for amalgamated products.

THEOREM 2.8. *Let X be a right denominator set in R such that $\tau_1(X) = \tau_2(X) = X$. Then*

- (i) *X is a right denominator set in S , and*
- (ii) *$SX^{-1} \cong (RX^{-1})[x; \tau_1, \delta_1] *_{RX^{-1}} (RX^{-1})[y; \tau_2, \delta_2]$,*

where $\tau_1, \tau_2, \delta_1, \delta_2$ denote the induced maps on the right ring of fractions RX^{-1} .

PROOF. Since X is a right denominator set in R , then the right ring of fractions RX^{-1} exists and we have

$$(RX^{-1})[x; \tau_1, \delta_1] *_{RX^{-1}} (RX^{-1})[y; \tau_2, \delta_2] = \sum_I RX^{-1} w_I.$$

In the amalgamated product ring $(RX^{-1})[x; \tau_1, \delta_1] *_{RX^{-1}} (RX^{-1})[y; \tau_2, \delta_2]$ we have the following identities for every $m \in X$:

$$\begin{aligned} m^{-1}x &= x(\tau_1^{-1}(m))^{-1} + m^{-1}(\delta_1\tau_1^{-1}(m))(\tau_1^{-1}(m))^{-1}, \\ m^{-1}y &= y(\tau_2^{-1}(m))^{-1} + m^{-1}(\delta_2\tau_2^{-1}(m))(\tau_2^{-1}(m))^{-1}. \end{aligned}$$

The two equations above show how m^{-1} commutes past the variables x and y . Thus, every element in $(RX^{-1})[x; \tau_1, \delta_1] *_{RX^{-1}} (RX^{-1})[y; \tau_2, \delta_2]$ can be written as a sum of the form $\sum_i r_i w_i m_i^{-1}$. Since X is a right Ore set in R , then for every $r_1, r_2 \in R$, $m_1, m_2 \in X$, and w_1, w_2 words in the letters x, y , there exist $r_3 \in R$ and $m_3 \in X$ so that

$$\begin{aligned} r_1 w_1 m_1^{-1} + r_2 w_2 m_2^{-1} &= (r_1 w_1 m_1^{-1} m_2 + r_2 w_2) m_2^{-1} \\ &= (r_1 w_1 r_3 m_3^{-1} + r_2 w_2) m_2^{-1} \\ &= (r_1 w_1 r_3 + r_2 w_2 m_3) m_3^{-1} m_2^{-1} \\ &= (r_1 w_1 r_3 + r_2 w_2 m_3) (m_2 m_3)^{-1} \in S(m_2 m_3)^{-1}. \end{aligned}$$

Therefore, right common denominators exist. Hence any $f \in (RX^{-1})[x; \tau_1, \delta_1] *_{RX^{-1}} (RX^{-1})[y; \tau_2, \delta_2]$ can be written in the form sm^{-1} for some $s \in S$ and $m \in X$. \square

We end this section by remarking that if R is a noetherian domain, then the set $X := R \setminus \{0\}$ satisfies the conditions in Theorem 2.8. In this case, the right ring of fractions SX^{-1} is isomorphic to an amalgamated product of Ore extensions over the division ring RX^{-1} .

2.2. Amalgamated Products of Quadratic Extensions. We call an extension $R \subseteq A$ of rings *quadratic* if there exists $x \in A \setminus R$ so that A is a free left R -module with basis $\{1, x\}$. Since $xR \subseteq Rx + R$, it follows that there exists a ring endomorphism $\tau : R \rightarrow R$ and a left τ -derivation $\delta : R \rightarrow R$ so that $xr = \tau(r)x + \delta(x)$ for every $r \in R$. Furthermore $x^2 = ax + b$ for some $a, b \in R$. Thus A is isomorphic to the factor ring $R[x; \tau, \delta] / \langle x^2 - ax - b \rangle$. However there are certain compatibility conditions involving the elements $a, b \in R$ and the endomorphism $\tau : R \rightarrow R$ that must hold. To make this precise we observe that for every $p \in R[x; \tau, \delta]$ there exist unique $r_0, r_1 \in R$ and $f \in R[x; \tau, \delta]$ so that $p = r_0 + r_1 x + f(x^2 - ax - b)$. This implies that for every $r \in R$, the following identities hold in $R[x; \tau, \delta]$:

$$(2.3) \quad (x^2 - ax - b)r = \tau^2(r)(x^2 - ax - b),$$

$$(2.4) \quad (x^2 - ax - b)x = (x + \tau(a) - a)(x^2 - ax - b).$$

If τ is an automorphism, Eqns. 2.3-2.4 are equivalent to $x^2 - ax - b$ being a normal element in $R[x; \tau, \delta]$.

Throughout this section, let Q_1 and Q_2 be arbitrary quadratic extensions of R . We will write them in the form $R[x; \tau_1, \delta_1]/I_1$ and $R[y; \tau_2, \delta_2]/I_2$ respectively, where $I_1 = \langle x^2 - ax - b \rangle$, $I_2 = \langle y^2 - cy - d \rangle$, and the triples (a, b, τ_1) and (c, d, τ_2) satisfy compatibility conditions analogous to those described in Eqns.2.3-2.4 above. For an element $f + I_i \in Q_i$ ($i = 1, 2$), we write it simply as f . Let $Q = Q_1 *_R Q_2$.

In this section, τ_1 and τ_2 do not necessarily need to be automorphisms of R . Unless stated otherwise, they are only assumed to be endomorphisms.

Let $\ell \geq 0$. We define the following alternating-letter words of length ℓ in Q :

$$x^{(\ell)} = xyxyx \cdots, \quad y^{(\ell)} = yxyxy \cdots, \quad \hat{x}^{(\ell)} = \cdots xyxyx, \quad \hat{y}^{(\ell)} = \cdots yxyxy.$$

THEOREM 2.9. *The ring Q is a free left R -module with basis $\{1, x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}, \dots\}$ (or equivalently $\{1, \hat{x}^{(1)}, \hat{y}^{(1)}, \hat{x}^{(2)}, \hat{y}^{(2)}, \dots\}$).*

PROOF. The relations $x^2 = ax + b$, $y^2 = cy + d$, $xr = \tau_1(r)x + \delta_1(r)$, $yr = \tau_2(r)y + \delta_2(r)$ ($r \in R$) imply that $\{1, x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}, \dots\}$ is a spanning set for Q . We need to prove R -linear independence. Let A be the free left R -module having basis $\{1, f_1, f_2, \dots, g_1, g_2, \dots\}$ and set $E := \text{End}_{\mathbb{Z}}(A)$. We view E as a left Q -module by first having each $r \in R$ act via left multiplication. For $p = c_0 + \sum_{i>0} (c_i f_i + d_i g_i) \in A$, we define the actions of x and y by

$$\begin{aligned} x.p &:= \delta_1(c_0) + \tau_1(c_1)b + \left(\tau_1(c_0) + \tau_1(c_1)a + \delta_1(c_1) \right) f_1 \\ &\quad + \sum_{i>0} \left(\tau_1(d_i) + \delta_1(c_{i+1}) + \tau_1(c_{i+1})a \right) f_{i+1} + \left(\delta_1(d_i) + \tau_1(c_{i+1})b \right) g_i, \\ y.p &:= \tau_2(d_1)d + \delta_2(c_0) + \sum_{i>0} \left(\delta_2(c_i) + \tau_2(d_{i+1})d \right) f_i + \left(\tau_2(c_{i-1}) + \tau_2(d_i)c + \delta_2(d_i) \right) g_i. \end{aligned}$$

One can verify that, as operators on E , we have $xr = \tau_1(r)x + \delta_1(r)$, $yr = \tau_2(r)y + \delta_2(r)$, $x^2 = ax + b$, and $y^2 = cy + d$. Therefore, these actions define a left Q -module structure on E . If $r_0 + \sum_{i>0} r_i x^{(i)} + r'_i y^{(i)} = 0 \in Q$ for some $r_i, r'_i \in R$, then $(r_0 + \sum_{i>0} r_i x^{(i)} + r'_i y^{(i)}) \cdot 1 = r_0 + \sum_{i>0} r_i f_i + r'_i g_i = 0$. This implies $r_0, r_1, r_2, \dots, r'_1, r'_2, \dots$ are all 0. Thus $1, x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}, \dots$ are left R -linearly independent in Q . \square

We remark that if τ_1 and τ_2 are automorphisms, then Q is also a free right R -module with basis $\{1, x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}, \dots\}$.

Next we prove an analogue of the Hilbert Basis theorem for Q . In the proof we make use of leading coefficients and leading terms. However, in contrast to an ordinary polynomial ring, some elements of Q can potentially have two leading coefficients instead of one. For instance if $p = \sum_{i=0}^n a_i x^{(i)} + b_i y^{(i)} \in Q$ ($a_i, b_i \in R$) with a_n, b_n not both zero, then we say p has degree n (or $\deg(p) = n$ for short). We call $a_n x^{(n)} + b_n y^{(n)}$ the *leading term* of p .

THEOREM 2.10. *If R is right (left) noetherian and τ_1, τ_2 automorphisms of R , then Q is right (left) noetherian.*

PROOF. First, let us suppose R is right noetherian. Let I be a right ideal of Q and define

$$\begin{aligned} L_1 &:= \{0\} \cup \{r \in R \mid \exists p \in I \text{ with leading term } rx^{(i)} \text{ for some } i \in \mathbb{N}\}, \\ L_2 &:= \{0\} \cup \{r \in R \mid \exists p \in I \text{ with leading term } a_i x^{(i)} + ry^{(i)} \text{ for some } a_i \in R, i \in \mathbb{N}\}, \end{aligned}$$

First, we show L_1 is a right ideal of R . The proof that L_2 is a right ideal of R is similar. Assume $\Lambda, \Lambda' \in L_1$ are nonzero. Thus, there exist $p_\Lambda, p_{\Lambda'} \in I$ having the form

$$\begin{aligned} p_\Lambda &= \Lambda x^{(i)} + [\text{lower degree terms}], \\ p_{\Lambda'} &= \Lambda' x^{(i')} + [\text{lower degree terms}]. \end{aligned}$$

Without any loss of generality, assume $i \leq i'$.

For every $m \geq \ell$, we let $x_{\ell,m}, y_{\ell,m} \in Q$ be the unique alternating-letter words of length $m - \ell$ that satisfy the conditions $x_{\ell,m}^{(\ell)} x_{\ell,m} = x^{(m)}$ and $y_{\ell,m}^{(\ell)} y_{\ell,m} = y^{(m)}$. If $\Lambda + \Lambda' = 0$, then $\Lambda + \Lambda' \in L_1$. On the other hand if $\Lambda + \Lambda' \neq 0$, then $p_{\Lambda} x_{i,i'} + p_{\Lambda'} \in I$ has leading term $(\Lambda + \Lambda') x^{(i')}$ and it follows that $\Lambda + \Lambda' \in L_1$. Furthermore, for any $r \in R$, we have $\Lambda r \in L_1$ because

$$p_{\Lambda}(\tau^{(i)})^{-1}(r) = \Lambda r x^{(i)} + [\text{lower degree terms}] \in I,$$

where $\tau^{(i)} = \underbrace{\tau_1 \tau_2 \tau_1 \tau_2 \tau_1 \cdots}_{i \text{ terms}}$. (We define $\tau^{(m)}$ similarly for every $m \in \mathbb{N}$.)

Therefore L_1 is a right ideal of R , hence finitely generated. Suppose $L_1 = r_1 R + \cdots + r_l R$ and $L_2 = s_1 R + \cdots + s_t R$. By multiplying on the right by appropriate words in x and y , we find that for every $1 \leq i \leq l$, $1 \leq j \leq t$, there exist $f_i, g_j \in I$ having the form

$$\begin{aligned} f_i &= r_i x^{(N)} + [\text{lower degree terms}] \\ g_j &= a_j x^{(N)} + s_j y^{(N)} + [\text{lower degree terms}] \end{aligned}$$

for some $N \in \mathbb{N}$ sufficiently large and some $a_j \in R$.

Next, define $M := R + x^{(1)}R + y^{(1)}R + \cdots + x^{(N-1)}R + y^{(N-1)}R$. Since M is a finitely generated right (and left) R -module, it follows that M is noetherian. Thus, the submodule $(I \cap M)_R \subseteq M_R$ is finitely generated. Suppose $I \cap M = c_1 R + \cdots + c_d R$.

Let I_0 be the right ideal of Q generated by $f_1, \dots, f_l, g_1, \dots, g_t, c_1, \dots, c_d$. We show $I = I_0$. From the definition of I_0 , it follows that $I_0 \subseteq I$. If $p \in I$ and $\deg(p) < N$, then $p \in I \cap M \subseteq I_0$. Thus, we suppose $p \in I$, $\deg(p) = m \geq N$, and everything in I having degree less than m is in I_0 . Let us assume

$$p = r x^{(m)} + s y^{(m)} + [\text{lower degree terms}] \in I$$

for some $r, s \in R$. Thus, there exist $b_1, \dots, b_t \in R$ so that $s = s_1 b_1 + \cdots + s_t b_t$. Next, put

$$u = \left(g_1 \left(\tau_1^{-1} \tau^{(N+1)} \right)^{-1} (b_1) + \cdots + g_t \left(\tau_1^{-1} \tau^{(N+1)} \right)^{-1} (b_t) \right) y_{N,m} \in I_0.$$

It follows that u has the form $u = r' x^{(m)} + s y^{(m)} + [\text{lower degree terms}]$ for some $r' \in R$. Therefore

$$p - u = (r - r') x^{(m)} + [\text{lower degree terms}].$$

Thus $r - r' = r_1 d_1 + \cdots + r_l d_l$ for some $d_1, \dots, d_l \in R$. Now define

$$u' := \left(f_1 \left(\tau^{(N)} \right)^{-1} (d_1) + \cdots + f_l \left(\tau^{(N)} \right)^{-1} (d_l) \right) x_{N,m} \in I_0.$$

It follows that $u' = (r - r') x^{(m)} + [\text{lower degree terms}]$. Therefore $p - u - u'$ has degree less than m , hence $p - u - u' \in I_0$. Therefore $p \in I_0$.

Now assume R is left noetherian. To prove Q is left noetherian, the argument is similar except now we construct left ideals L_1, L_2 of R by writing polynomials in Q with *right-hand* coefficients and use the *right* R -basis $\{1, \hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \dots\}$ of Q_R . \square

DEFINITION 2.11. Let τ be an endomorphism of a ring R . A left τ -derivation δ is inner if there exists $s \in R$ so that $\delta(r) = \tau(r)s - sr$ for all $r \in R$.

THEOREM 2.12. Suppose

- (i) R is a division ring,
- (ii) τ_1 and τ_2 are automorphisms, and
- (iii) neither δ_i is an inner τ_i -derivation.

Then Q is a principal ideal ring.

PROOF. Let n be the minimal degree among the nonzero elements of a nonzero proper ideal $I \subseteq Q$ and suppose

$$f = x^{(n)} + a_{n-1} x^{(n-1)} + b_{n-1} y^{(n-1)} + [\text{lower degree terms}] \in I$$

for some $a_{n-1}, b_{n-1} \in R$. Then for all $r \in R$, we compute

$$\begin{aligned} fr - \tau^{(n)}(r)f &= \left(\tau^{(n-1)}\delta_{[[n]]}(r) + a_{n-1}\tau^{(n-1)}(r) - \tau^{(n)}(r)a_{n-1} \right) x^{(n-1)} \\ &\quad + \left(\delta_1\tau_1^{-1}\tau^{(n)}(r) + b_{n-1}\tau_1^{-1}\tau^{(n)}(r) - \tau^{(n)}(r)b_{n-1} \right) y^{(n-1)} \\ &\quad + [\text{lower degree terms}] \end{aligned}$$

where $[[n]] = 1$ (resp. 2) when n is odd (resp. even), and $\tau^{(i)} = \underbrace{\tau_1\tau_2\tau_1\cdots}_{i \text{ terms}}$ for all $i \in \mathbb{N}$.

Since $\deg(fr - \tau^{(n)}(r)f) < n$ and $fr - \tau^{(n)}(r)f \in I$, this implies the coefficients above are all zero. In particular, this will imply that δ_1 is an inner τ_1 -derivation (because $\delta_1(r) = \tau_1(r)b_{n-1} - b_{n-1}r$ for all $r \in R$). Thus $f \notin I$. Similarly, one can show that I does not contain anything of the form $y^{(n)} + [\text{lower degree terms}]$.

Let I_n denote the set of elements of I having degree n . From the previous arguments, it follows that every $g \in I_n$ has both leading coefficients nonzero. Hence, there exists $p = a_n x^{(n)} + b_n y^{(n)} + [\text{lower terms}] \in I$ with $a_n, b_n \in R$ both nonzero. If $h = c_n x^{(n)} + d_n y^{(n)} + [\text{lower terms}] \in I_n$ for some nonzero $c_n, d_n \in R$, then

$$a_n^{-1}p - c_n^{-1}h = (a_n^{-1}b_n - c_n^{-1}d_n)y^{(n)} + [\text{lower terms}] \in I.$$

Therefore $a_n^{-1}b_n - c_n^{-1}d_n = 0$. Hence, we have

$$a_n c_n^{-1}h = a_n x^{(n)} + b_n y^{(n)} + [\text{lower terms}] \in I_n.$$

It follows that $\deg(p - a_n c_n^{-1}h) < n$. Hence $p - a_n c_n^{-1}h = 0$ and $I_n = (R \setminus \{0\})p$.

We will show $I = \langle p \rangle$. Obviously, $\langle p \rangle \subseteq I$. Furthermore, if $p' \in I$ and $\deg(p') < n$, then $p' = 0$. Thus $p' \in \langle p \rangle$. If $\deg(p') = n$, then $p' \in Rp \subseteq \langle p \rangle$. Now suppose $p' = a'_m x^{(m)} + b'_m y^{(m)} + [\text{lower degree terms}] \in I$ with $m > n$ and assume everything in I having degree less than m is in $\langle p \rangle$. Since

$$\deg(p' - a'_m a_n^{-1} p x_{n,m} - b'_m b_n^{-1} p y_{n,m}) < m,$$

(recall that $x_{n,m}, y_{n,m} \in Q$ are the unique alternating-letter words of length $m - n$ so that $x^{(n)}x_{n,m} = x^{(m)}$ and $y^{(n)}y_{n,m} = y^{(m)}$) this implies $p' \in \langle p \rangle$. \square

THEOREM 2.13. *If R is a division ring and τ_1 and τ_2 are automorphisms, then the one-sided ideals of Q are either principal or doubly generated.*

PROOF. We will prove the left ideals of Q are either principal or doubly generated. Here, we use the fact that Q is a free left R -module having basis $\{1, \hat{x}^{(1)}, \hat{y}^{(1)}, \hat{x}^{(2)}, \hat{y}^{(2)}, \dots\}$ and write all polynomials with left-hand coefficients. The proof for the right ideals is similar; write all polynomials with right-hand coefficients. Suppose I is a nonzero left ideal of Q . Let n be the minimal degree among the nonzero elements of I . Choose $p \in I$ having degree n . We consider three cases. In all cases we show there exists $p' \in I$ so that $I = Qp + Qp'$. Whenever I is a principal left ideal, p' may be chosen to be in Qp . In this situation, we do not mention p' . We let f be an arbitrary element of I having degree m . If $m < n$, then $f = 0$ and clearly $f \in Qp + Qp'$. Thus, we let $m \geq n$ and assume everything in I of degree less than m is in $Qp + Qp'$. Throughout this proof, we will make use of elements $\hat{x}_{n,m}, \hat{y}_{n,m} \in Q$ (for $m \geq n$), which are defined by the conditions $\hat{x}_{n,m}\hat{x}^{(n)} = \hat{x}^{(m)}$ and $\hat{y}_{n,m}\hat{y}^{(n)} = \hat{y}^{(m)}$.

Case I: Every nonzero element of I has no leading $\hat{x}^{(i)}$ -coefficients ($\hat{y}^{(i)}$ -coefficients). Since R is a division ring, we may without any loss of generality assume the leading term of p is $\hat{y}^{(n)}$ (or $\hat{x}^{(n)}$). Suppose the leading term of f is $r\hat{y}^{(m)}$ (or $r\hat{x}^{(m)}$) for some nonzero $r \in R$. Hence $\deg[f - r(\hat{x}_{n,m} + \hat{y}_{n,m})p] < m$. Therefore, $I = Qp$.

Case II: Every element in I of degree n has no leading $\hat{x}^{(i)}$ -coefficient ($\hat{y}^{(i)}$ -coefficient) and there exists $p' \in I$ having a nonzero leading $\hat{x}^{(i)}$ -coefficient ($\hat{y}^{(i)}$ -coefficient). We will prove this for the $\hat{x}^{(i)}$ -case. In other words, we assume that every polynomial in I of degree n has 0 as its leading $\hat{x}^{(i)}$ -coefficient and there exists $p' \in I$ having a nonzero leading $\hat{x}^{(i)}$ -coefficient. The proof for the $\hat{y}^{(i)}$ -case is similar. Choose $p' \in I$

of minimal degree satisfying the aforementioned condition. Without loss of generality, suppose

$$\begin{aligned} p &= \widehat{y}^{(n)} + [\text{lower degree terms}], \\ p' &= \widehat{x}^{(l)} + r\widehat{y}^{(l)} + [\text{lower degree terms}], \\ f &= s\widehat{x}^{(m)} + t\widehat{y}^{(m)} + [\text{lower degree terms}] \end{aligned}$$

for some $r, s, t \in R$. It readily follows that $f - t\widehat{y}_{n,m}p = s\widehat{x}^{(m)} + [\text{lower degree terms}]$. If $s = 0$, then $\deg(f - t\widehat{y}_{n,m}p) < m$ and this implies $f \in Qp + Qp'$. If $s \neq 0$, then $m \geq l$ and $\deg(f - t\widehat{y}_{n,m}p - s\widehat{x}_{l,m}p') < m$. Thus $f \in Qp + Qp'$.

Case III: p may be chosen with both leading coefficients nonzero. Suppose without loss of generality

$$p = \widehat{x}^{(n)} + r\widehat{y}^{(n)} + [\text{lower degree terms}]$$

for some nonzero $r \in R$. Let $p' \in I$ be a nonzero polynomial of minimal degree having 0 as its leading $\widehat{x}^{(i)}$ -coefficient (such a p' exists because $\widehat{y}_{n,n+1}r^{-1}p = \widehat{y}^{(n+1)} + [\text{lower degree terms}] \in I$, for instance). Since R is a division ring, we may choose p' to be of the form $p' = \widehat{y}^{(l)} + [\text{lower degree terms}]$. Suppose $f = s\widehat{x}^{(m)} + t\widehat{y}^{(m)} + [\text{lower degree terms}]$ for some $s, t \in R$. The leading term of $f - s\widehat{x}_{n,m}p$ is $t\widehat{y}^{(m)}$. If $t = 0$, then $\deg(f - s\widehat{x}_{n,m}p) < m$. Therefore $f \in Qp + Qp'$. If $t \neq 0$, then $m \geq l$ and $\deg(f - s\widehat{x}_{n,m}p - t\widehat{y}_{l,m}p') < m$. Thus $f \in Qp + Qp'$. \square

3. The Double Affine Hecke Algebra of Type GL_2 : An Example of an Amalgamated Product

In this section we show that the double affine Hecke algebra $\mathbb{H}_{q,t}(GL_2(k))$ associated to the general linear group $GL_2(k)$ is an amalgamated product of quadratic extensions over a three dimensional quantum torus provided $\text{char}(k) \neq 2$. From the results of Section 2.2 it follows that $\mathbb{H}_{q,t}(GL_2(k))$ is noetherian.

The presentation we use for $\mathbb{H}_{q,t}(GL_2(k))$ is taken from [2, Section 1.4.3]. The algebra $\mathbb{H}_{q,t}(GL_2(k))$ can be defined over any base field k . However, in the results that follow we need $\text{char}(k) \neq 2$ because 2 appears in a denominator. Let $q, t \in k$ be fixed nonzero scalars such that $t^{1/2}$ exists. The double affine Hecke algebra $\mathbb{H}_{q,t}(GL_2(k))$ is defined as the associative k -algebra generated by invertible elements T, X_1, X_2, Y_1, Y_2 and has the defining relations

$$\begin{aligned} (3.1) \quad X_1X_2 &= X_2X_1, & Y_1Y_2 &= Y_2Y_1, \\ (3.2) \quad (T - t^{1/2})(T + t^{-1/2}) &= 0, & Y_2^{-1}X_1Y_2X_1^{-1} &= T^2, \\ (3.3) \quad T^{-1}Y_1T^{-1} &= Y_2, & TX_1T &= X_2, \\ (3.4) \quad Y_1Y_2X_1 &= qX_1Y_1Y_2, & Y_1Y_2X_2 &= qX_2Y_1Y_2, \\ (3.5) \quad X_1X_2Y_1 &= q^{-1}Y_1X_1X_2, & X_1X_2Y_2 &= q^{-1}Y_2X_1X_2. \end{aligned}$$

Let R denote the k -algebra generated by the variables $z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}$ and having the defining relations $z_1z_2 = z_2z_1$, $z_1z_3 = q^{-1}z_3z_1$, and $z_2z_3 = q^{-1}z_3z_2$. Let τ_1 be the k -algebra automorphism of R that interchanges z_1 and z_2 , and sends z_3 to itself. Finally, let δ_1 be the k -linear left τ_1 -derivation of R defined by $-\alpha \frac{z_1+z_2}{z_1-z_2}(1-\tau_1)$, where $\alpha = \frac{1}{2}(t^{1/2} - t^{-1/2}) \in k$. Put $Q_1 = R[x; \tau_1, \delta_1]/I_1$, where $I_1 \subseteq R[x; \tau_1, \delta_1]$ is the ideal generated by the normal element $x^2 - \left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^2$. Therefore Q_1 is a quadratic extension of R . When it is not confusing, we will let x denote the equivalency class $x + I_1 \in Q_1$. Next, let $Q_2 = R[y; \tau_2, \delta_2]/I_2$, where τ_2 is the automorphism of R given by $z_1 \mapsto z_2, z_2 \mapsto q^{-1}z_1, z_3 \mapsto z_3, \delta_2 \equiv 0$, and $I_2 \subseteq R[y; \tau_2, \delta_2]$ is the ideal generated by the normal element $y^2 - z_3^{-1}$. Thus Q_2 is a quadratic extension of R . Let y denote the equivalency class $y + I_2 \in Q_2$.

Let $Q = Q_1 *_R Q_2$. We have the following

THEOREM 3.1. *(for $\text{char}(k) \neq 2$) There is a k -algebra isomorphism $\varphi : Q \rightarrow \mathbb{H}_{q,t}(GL_2(k))$ which sends the generators of Q to the following:*

$$z_1 \mapsto X_1, \quad z_2 \mapsto X_2, \quad z_3 \mapsto Y_1Y_2, \quad x \mapsto T - \alpha, \quad y \mapsto Y_1^{-1}T,$$

where $\alpha = \frac{1}{2}(t^{1/2} - t^{-1/2})$.

PROOF. It is straightforward to check that the map φ above defines an algebra homomorphism. To show φ is an isomorphism, we note that there is an algebra homomorphism $\tilde{\varphi} : \mathbb{H}_{q,t}(GL_2(k)) \rightarrow Q$ given by $X_1 \mapsto z_1$, $X_2 \mapsto z_2$, $Y_1 \mapsto z_3(x + \alpha)y$, $Y_2 \mapsto z_3y(x - \alpha)$, and $T \mapsto x + \alpha$. Finally, one can verify that $\varphi\tilde{\varphi} = \text{Id}_{\mathbb{H}_{q,t}(GL_2(k))}$ and $\tilde{\varphi}\varphi = \text{Id}_Q$. \square

Therefore, we have the following

THEOREM 3.2. (for $\text{char}(k) \neq 2$) *The double affine Hecke algebra $\mathbb{H}_{q,t}(GL_2(k))$ is a noetherian ring.*

PROOF. The quantum torus R is a noetherian ring. By Theorems 2.10 and 3.1, $\mathbb{H}_{q,t}(GL_2(k))$ is noetherian also. \square

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